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Pseudo Invariant Eigenoperator Method for Some Generalized Jaynes-Cummings Models

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Abstract Based on the method of pseudo invariant eigenoperator (PIEO), we investigate three kinds of the generalized Jaynes-Cummings (JC) models such as the super JC model, the Kerr nonlinear JC model, and the two-atomic two-photon JC model. Our main task lies in finding the so-called pseudo invariant eigenoperators and deriving the energy-level gap for the above Hamiltonians, respectively. Compared with the usual Schrodinger equation approach or the directly diagonalizing Hamiltonian, the PIEO method could be quite concise and effective to obtain energy-level gap of the given system.

Keywords Pseudo invariant eigenoperator · Energy gap · Jaynes-Cummings model

1 Introduction

It is well known that the Jaynes-Cummings (JC) model [1] plays an important role in the study of a single two-level atom interacting with a single quantized electromagnetic fields, which is shown to exhibit interesting nonclassical effects, such as the collapse and revival of the Rabi oscillations of the atomic inversion, antibunched light, and squeezing. Over the past many years, various exactly solvable modifications and generalizations of the original JC model have been studied widely such as the Kerr effect, the Stark effect, the Doppler effect, and so on [2–6]. In the investigations of these models, it is a basic task to obtain their energy spectra and eigenstates. Therefore, developing simple and efficient approaches for solving energy eigenvalue problems is very significant for both theoretical interest and its wide applications. Originally, the JC model was diagonalized by Carbonaro et al. [7]. In 1996, Fan et al. [8] also diagonalized the JC Hamiltonian via supersymmetric generators. Subsequently, some authors further have applied this method to calculate the eigenvalues

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and eigenstates for other some JC models [9–13]. Xu et al. [14] use a dynamical algebraic method to obtain the solutions for the modified JC Hamiltonian. Recently, in [15–19], they have developed invariant eigenoperators (IEO) theory and further extended the pseudo invariant eigenoperators (PIEO) method to derive the energy spectrum of some generalized JC Hamiltonians [20, 21].

In this work, based on the above ideas, we further use the PIEO method to discuss some generalized JC models including the super JC model, the Kerr nonlinear JC model, and the two-atomic two-photon JC model, respectively. Our main task is to search for the so-called pseudo-invariant eigenoperators and then derive the energy-level gap for the above systems. The results show that the PIEO method could be simpler than the usual Schrödinger equation approach or the directly diagonalizing Hamiltonians approach so far as obtaining energy-level gap is concerned. Finally, the paper is concluded with a summary and some discussions.

2 Pseudo Invariant Eigenoperator for the Super Jaynes-Cummings Model

Under the rotating-wave approximation, the considered Hamiltonian of the super Jaynes-Cummings model [22] is given by (setting $\hbar = 1$)

$$H_1 = 2\omega_1 a^\dagger a + 2\omega_2 f^\dagger f + \chi a f^\dagger + a^\dagger f \overline{\chi}, \tag{1}$$

in which the coupling constants χ and $\overline{\chi}$ are considered as Grassmann value, i.e. $\chi^2 = \overline{\chi}^2 = 0$, $\{\chi, f\} = 0$. Here, $a(a^\dagger)$ and $f(f^\dagger)$ are the creation (annihilation) operator of boson and fermion, respectively, satisfying the superalgebra

$$[a, a^\dagger] = 1, \qquad \{f, f^\dagger\} = 1, \qquad [a, f] = [a^\dagger, f] = 0, \qquad f^2 = f^{\dagger 2} = 0. \tag{2}$$

ω_1 and ω_2 are the frequency of the bosonic mode and fermionic mode, respectively.

Next, we find pseudo invariant eigenoperator for the super JC model. Let $|0\rangle$ be a vacuum state of fermion, then $f|0\rangle = 0$, $f^\dagger|0\rangle = |1\rangle$, we have

$$f = |0\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad f^\dagger = |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3}$$

and

$$f^\dagger f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \equiv \frac{1}{2}(\sigma_z' + 1) \tag{4}$$

where $|0\rangle$ and $|1\rangle$ denote the matrix $\binom{0}{1}$ and $\binom{1}{0}$, respectively, σ_z' represents the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In order to redescribe the super JC Hamiltonian via the supersymmetric operator, we first denote the supersymmetric generators as

$$\begin{aligned} Q_1 &= a^\dagger f = \begin{pmatrix} 0 & 0 \\ a^\dagger & 0 \end{pmatrix}, \\ Q_1^\dagger &= f^\dagger a = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \\ N_1' &= a^\dagger a + f^\dagger f = \begin{pmatrix} aa^\dagger & 0 \\ 0 & a^\dagger a \end{pmatrix}. \end{aligned} \tag{5}$$

It is easily seen that N_1', Q_1, $Q_1^\dagger$ form the supersymmetric generators and have Lie superalgebra properties, i.e.,

$$\begin{aligned} &Q_1^2 = Q_1^{\dagger 2} = 0, \qquad [Q_1^\dagger, Q_1] = N_1'\sigma_z', \qquad (Q_1^\dagger - Q_1)^2 = -N_1', \\ &\left[N_1', Q_1\right] = [N_1', Q_1^\dagger] = 0, \qquad \left[Q_1, \sigma_z'\right] = 2Q_1, \qquad [Q_1^\dagger, \sigma_z'] = -2Q_1^\dagger, \\ &\left\{Q_1, \sigma_z'\right\} = \{Q_1^\dagger, \sigma_z'\} = 0, \qquad \{Q_1, Q_1^\dagger\} = N_1'. \end{aligned} \tag{6}$$

In terms of the above generators, we can rewrite the Hamiltonian equation (1) as

$$H_1 = 2\omega_1 N' + (\omega_2 - \omega_1)\sigma_z' + \chi Q_1^\dagger + Q_1\overline{\chi} + (\omega_2 - \omega_1). \tag{7}$$

From the Hamiltonian in (7), it is easily seen that H_1 is a linear combination of the generators N_1', Q_1, $Q_1^\dagger$. Hence, according to the PIEO theory (see Appendix in detail) [20, 21], we assume that the PIEO of H_1 has the following form

$$\hat{O}_{1e} = \alpha_1(Q_1^\dagger + Q_1) + \beta_1\sigma_z', \tag{8}$$

where α_1 and β_1 are two real constants to be undermined. With the help of the relations in (6), we calculate

$$i\frac{d}{dt}\hat{O}_{1e} = [\hat{O}_{1e}, \hat{H}_1] = 2[\alpha_1\Delta_1 - \beta_1(\chi + \overline{\chi})](Q_1 - Q_1^\dagger). \tag{9}$$

Further calculation shows

$$\begin{aligned} \left(i\frac{d}{dt}\right)^2 \hat{O}_{1e} &= [[\hat{O}_{1e}, \hat{H}_1], \hat{H}_1] \\ &= 2[\alpha_1\Delta_1 - \beta_1(\chi + \overline{\chi})][2\alpha_1\Delta_1(Q_1^\dagger + Q_1) - (\chi + \overline{\chi})N_1'\sigma_z'], \end{aligned} \tag{10}$$

where $\Delta_1 = (\omega_2 - \omega_1)$. Comparing the right hand side of (10) with (8), we see that $2\alpha_1\Delta_1(Q_1^\dagger + Q_1) - (\chi + \overline{\chi})N_1'\sigma_z'$ differs from the structure of $\hat{O}_{1e}$ since $N_1'\sigma_z'$ differs from σ_z'. Thus, it is unlikely that $\hat{O}_{1e}$ can satisfy the eigenoperator equation for the $n = 2$ case in (A.2) of Appendix. However, we may understand the (10) in the sense of eigenvalue and eigenvalue equations for the operator N_1', i.e.

$$\left|\psi'\right\rangle_1 = |n\rangle \otimes |1\rangle = \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}, \qquad \left|\psi'\right\rangle_2 = |n+1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ |n+1\rangle \end{pmatrix}, \tag{11}$$

$$N_1'\left|\psi'\right\rangle_i = (n+1)\left|\psi'\right\rangle_i, \quad i = 1, 2. \tag{12}$$

From (12), by acting the two sides of (10) on the eigenstates $|\psi'\rangle_i$ in the (11), we have

$$\left(i\frac{d}{dt}\right)^2 \hat{O}_{1e}\left|\psi'\right\rangle_i = 2[\alpha_1\Delta_1 - \beta_1(\chi + \overline{\chi})][2\Delta_1(Q_1^\dagger + Q_1) - (\chi + \overline{\chi})(n+1)\sigma_z']\left|\psi'\right\rangle_i, \tag{13}$$

which is a form like (A.3) in Appendix. Thus we can compare (8) and $2\Delta_1(Q_1^\dagger + Q_1) - (\chi + \overline{\chi})(n+1)\sigma_z'$ and find

$$\alpha_1 : \beta_1 = 2\Delta_1 : [-(\chi + \overline{\chi})(n+1)]. \tag{14}$$

It then follows that

$$\alpha_1 = -\frac{2\Delta_1}{(\chi + \overline{\chi})(n+1)}\beta_1 \tag{15}$$

and (8) becomes

$$\hat{O}_{1e} = -\frac{2\Delta_1}{(\chi + \overline{\chi})(n+1)}\beta_1(Q^\dagger + Q) + \beta_1\sigma_z', \tag{16}$$

which is just our finding pseudo invariant eigenoperator for (1). From the right-hand side of (13) and $(i\frac{d}{dt})^2\hat{O}_{1e} = \lambda_1\hat{O}_{1e}$, we readily deduce

$$\lambda_1 = 4[\Delta_1^2 + \chi\overline{\chi}(n+1)] \tag{17}$$

and further obtain the energy-level gap for this system in (1) as

$$\sqrt{\lambda_1} = 2\sqrt{\Delta_1^2 + \chi\overline{\chi}(n+1)}. \tag{18}$$

3 Pseudo Invariant Eigenoperator for the Jaynes-Cummings Model with Kerr Nonlinearity

In this section, we search for the PIEO for the JC model including the Kerr nonlinearity, which reads as [23]

$$H_2 = \omega a^\dagger a + \frac{1}{2}\omega_0\sigma_z + q a^{\dagger 2}a^2 + g[a^\dagger\sigma_- + a\sigma_+], \tag{19}$$

where ω and ω_0 are the field and atomic transition frequencies, respectively; $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the two-level atomic inversion operator, $\sigma_+ = |+\rangle\langle-|$ and $\sigma_- = |-\rangle\langle+|$ are the atomic raising and lowering operators, respectively, with $|-\rangle = \binom{0}{1}(|+\rangle = \binom{1}{0})$ denoting the atomic ground state (atomic excited state). Here g is the atom-field coupling constant and q represents the nonlinear parameter of the Kerr medium.

Similarly, we introduce its supersymmetric generators as follows,

$$\begin{aligned} Q_2 &= a^\dagger\sigma_- = \begin{pmatrix} 0 & 0 \\ a^\dagger & 0 \end{pmatrix}, \\ Q_2^\dagger &= a\sigma_+ = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \\ N_2' &= a^\dagger a\sigma_-\sigma_+ + aa^\dagger\sigma_+\sigma_- \begin{pmatrix} aa^\dagger & 0 \\ 0 & a^\dagger a \end{pmatrix}, \end{aligned} \tag{20}$$

where

$$\sigma_{++} \equiv \sigma_+\sigma_- = \frac{1}{2}(1+\sigma_z), \sigma_{--} \equiv \sigma_-\sigma_+ = \frac{1}{2}(1-\sigma_z), \tag{21}$$

which obey the commutation and anticommutation relations,

$$\begin{aligned} &Q_2^2 = Q_2^{\dagger 2} = 0, \qquad [Q_2^\dagger, Q_2] = N_2'\sigma_z, \qquad (Q_2^\dagger - Q_2)^2 = -N_2', \\ &[N_2', Q_2] = \left[N_2', Q_2^\dagger\right] = 0, \qquad [Q_2, \sigma_z] = 2Q_2, \qquad [Q_2^\dagger, \sigma_z] = -2Q_2^\dagger, \\ &\{Q_2, \sigma_z\} = \{Q_2^\dagger, \sigma_z\} = 0, \qquad \{Q_2, Q_2^\dagger\} = N_2'. \end{aligned} \tag{22}$$

It is easily proved that $N_2' = a^\dagger a + \frac{1}{2}(1+\sigma_z)$ is constant of motion. Using the property $\sigma_z^2 = 1$, we obtain that

$$a^{\dagger 2} a^2 = a^\dagger a(a^\dagger a - 1) = (N_2' - 1)^2 - (N_2' - 1)\sigma_z. \tag{23}$$

With the help of (20) and (23), the Hamiltonian H_2 may be rewritten as

$$H_2 = H_{20} + \frac{1}{2}\Delta_2(N_2')\sigma_z + g(Q_2 + Q_2^\dagger), \tag{24}$$

where

$$H_{20} = N_2'\omega - \frac{1}{2}\omega + (N_2' - 1)^2, \qquad \Delta_2(N_2') = -2(N_2' - 1) + \omega_0 - \omega. \tag{25}$$

We assume that the PIEO of H_2 is possessed of the form

$$\hat{O}_{2e} = \alpha_2(Q_2^\dagger + Q_2) + \beta_2\sigma_z, \tag{26}$$

where α_2 and β_2 are undermined constants. Via the relations in (22), it is calculated

$$\left(i\frac{d}{dt}\right)^2 \hat{O}_{2e} = [[\hat{O}_{2e}, \hat{H}_2], \hat{H}_2] = (\alpha_2\Delta_2(N_2') - 2\beta_2 g)[\Delta_2(N_2')(Q_2^\dagger + Q_2) - 2gN_2'\sigma_z]. \tag{27}$$

According to the PIEO theory [20, 21], we act the two sides of (27) on the eigenstates $|\psi''\rangle_i$ for the operators N_2', i.e.

$$\left|\psi''\right\rangle_1 = |n\rangle \otimes |+\rangle = \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}, \qquad \left|\psi''\right\rangle_2 = |n+1\rangle \otimes |-\rangle = \begin{pmatrix} 0 \\ |n+1\rangle \end{pmatrix}, \tag{28}$$

$$N_2'\left|\psi''\right\rangle_i = (n+1)\left|\psi''\right\rangle_i, \quad i = 1, 2, \tag{29}$$

where $\sigma_z|\pm\rangle = \pm 1|\pm\rangle$ and further deduce

$$\left(i\frac{d}{dt}\right)^2 \hat{O}_e\,|\psi\rangle_i = (\alpha_2\Delta_2(n+1) - 2\beta_2 g)[\Delta_2(n+1)(Q_2^\dagger + Q_2) - 2g(n+1)\sigma_z]\left|\psi''\right\rangle_i. \tag{30}$$

From (26) and (30), we obtain

$$\alpha_2 = -\frac{\Delta_2(n+1)}{(n+1)\cdot 2g}\beta_2, \tag{31}$$

where $\Delta_2(n+1) = -2n + \omega_0 - \omega$. Thus in the Hilbert space spanned by the eigenstates $|\psi''\rangle_i$, we may determine the expression of $\hat{O}_{2e}$

$$\hat{O}_{2e} = -\frac{\Delta_2(n+1)}{(n+1)\cdot 2g}\beta_2(Q_2^\dagger + Q_2) + \beta_2\sigma_z. \tag{32}$$

Substituting (31) and (32) into $(i\frac{d}{dt})^2\hat{O}_{2e} = \lambda_2\hat{O}_{2e}$, we get

$$\lambda_2 = \Delta_2^2(n+1) + 4g^2(n+1), \tag{33}$$

so the energy-level gap for the model H_2 is

$$\sqrt{\lambda_2} = \sqrt{\Delta_2^2(n+1) + 4g^2(n+1)}. \tag{34}$$

4 Pseudo Invariant Eigenoperator for the Jaynes-Cummings Model with Two-Atomic Two-Photon Process

In the following section, by utilizing the similar method we investigate the JC model with two-atomic two-photon process [12]. Its physical meaning is that the separation between the two atoms is much less than the ware length of the fields and each atom emits or absorbs two photons during the interaction, whose Hamiltonian is,

$$H_3 = \omega(a_1^\dagger a_1 + a_2^\dagger a_2) + \frac{1}{2}\omega_0(\sigma_{1z} + \sigma_{2z}) + g[a_1^{\dagger 2} a_2^{\dagger 2}\sigma_{1-}\sigma_{2-} + a_1^2 a_2^2 \sigma_{1+}\sigma_{2+}]. \tag{35}$$

To begin with, we also introduce the matrix of the supersymmetric generators

$$Q_3 = a_1^{\dagger 2} a_2^{\dagger 2}\sigma_{1-} \otimes \sigma_{2-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1^{\dagger 2} a_2^{\dagger 2} & 0 & 0 & 0 \end{pmatrix},$$

$$Q_3^\dagger = a_1^2 a_2^2 \sigma_{1+} \otimes \sigma_{2+} = \begin{pmatrix} 0 & 0 & 0 & a_1^2 a_2^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

$$N_3' = \begin{pmatrix} a_1^2 a_2^2 a_1^{\dagger 2} a_2^{\dagger 2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1^{\dagger 2} a_2^{\dagger 2} a_1^2 a_2^2 \end{pmatrix}.$$

It is easily proved that N_3', Q_3, $Q_3^\dagger$ is the supersymmetric generators as well, i.e.

$$\begin{gathered} Q_3^2 = Q_3^{\dagger 2} = 0, \qquad \left[Q_3^\dagger, Q_3\right] = \frac{1}{2} N_3' \tilde{\sigma}_{z3}, \qquad \left(Q_3^\dagger - Q_3\right)^2 = -N_3', \\ \left[N_3', Q_3\right] = \left[N_3', Q_3^\dagger\right] = 0, \qquad \left[Q_3, \tilde{\sigma}_{z3}\right] = 4Q_3, \qquad \left[Q_3^\dagger, \tilde{\sigma}_{z3}\right] = -4Q_3^\dagger, \\ \{Q, \tilde{\sigma}_z\} = \left\{Q^\dagger, \tilde{\sigma}_{z3}\right\} = 0, \qquad \left\{Q_3, Q_3^\dagger\right\} = N_3', \end{gathered} \tag{37}$$

where

$$\tilde{\sigma}_{z3} \equiv \sigma_{1z} \otimes I_2 + \sigma_{2z} \otimes I_1 = 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{38}$$

We may use (36) and rewrite the Hamiltonian H_3 as

$$H_3 = \omega M_3 + \frac{1}{2}\Delta_3 \tilde{\sigma}_{z3} + g(Q_3 + Q_3^\dagger) \tag{39}$$

where $\Delta_3 = \Omega - 2\omega$ is the frequency detuning and $M_3 = a_1^\dagger a_1 + a_2^\dagger a_2 + \tilde{\sigma}_{z3}$ is constant of motion obeying the commutive relation

$$\left[M_3, N_3'\right] = [M_3, Q_3] = [M_3, Q_3^\dagger] = 0. \tag{40}$$

Next, let the form of $\hat{O}_{3e}$ for the Hamilton $\hat{H}_3$,

$$\hat{O}_{3e} = \alpha_3(Q_3^\dagger + Q_3) + \beta_3\tilde{\sigma}_{z3}, \tag{41}$$

we obtain the following result

$$\left(i\frac{d}{dt}\right)^2 \hat{O}_{3e} = 2(\alpha_3\Delta_3 - 2\beta_3 g)[2\Delta_3(Q_3^\dagger + Q_3) - gN_3'\tilde{\sigma}_{z3}]. \tag{42}$$

Similarly, we may deduce the expression of $\hat{O}_{3e}$

$$\hat{O}_{3e} = -\frac{2\Delta_3}{gG_3(n_1,n_2)}\beta_3(Q^\dagger + Q) + \beta_3\sigma_z \tag{43}$$

in the Hilbert space spanned by the eigenstates $|\psi'''\rangle_i$ of the operators M_3 and N_3', i.e.

$$\left|\psi'''\right\rangle_1 = |n_1\rangle_1 \otimes |+\rangle_1 \otimes |n_2\rangle_2 \otimes |+\rangle_2 = \begin{pmatrix} |n_1\rangle_1\,|n_2\rangle_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{44}$$

$$\left|\psi'''\right\rangle_2 = |n_1+2\rangle_1 \otimes |+\rangle_1 \otimes |n_2+2\rangle_2 \otimes |+\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ |n_1+2\rangle_1\,|n_2+2\rangle_2 \end{pmatrix}, \tag{45}$$

$$N_3'\left|\psi'''\right\rangle_i = G_3(n_1,n_2)\left|\psi'''\right\rangle_i, \quad i = 1,2, \tag{46}$$

$$M\left|\psi'''\right\rangle_i = (n_1+n_2+2)\left|\psi'''\right\rangle_i, \quad i = 1,2, \tag{47}$$

where $G_3(n_1,n_2) \equiv (n_1+1)(n_1+2)(n_2+1)(n_2+2)$. Onceagain, considering $(i\frac{d}{dt})^2\hat{O}_{3e} = \lambda_3\hat{O}_{3e}$, we obtain

$$\lambda_3 = 4[\Delta_3^2 + g^2G_3(n_1,n_2)] \tag{48}$$

and further educe the energy-level gap for the Hamiltonian H_3 as

$$\sqrt{\lambda_3} = 2\sqrt{\Delta_3^2 + g^2G_3(n_1,n_2)}, \tag{49}$$

which coincides with the eigen-energy of H_3 discussed by [12] via supersymmetric unitary transformation.

5 Discussion and Summary

In summary, by virtue of the PIEO method we have found the pseudo invariant eigenoperators in terms of supersymmetric generators to easily lead to the energy-level gaps for the super JC model, the Kerr nonlinear JC model, and the two-atomic two-photon JC model, respectively. As one can see in the above discussions, the essential point in order to proceed is to find the PIEO for different JC Hamiltonians, this can be started by examining some fundamental commutative relations between basic operators with the Hamiltonian, and then determining the energy quantization condition. Compared with [12, 21, 22], the

PIEO method could be simpler than the usual Schrodinger equation approach or the directly diagonalizing Hamiltonian approach so far as obtaining energy-level gap is concerned. Because the Schrödinger equation always leads to differential equations which in many cases are difficult to solve. Finally, We believe that this new approach may be extended to dealing with more other generalized JC models, provided that the PIEO can be found.

Appendix

In this appendix, we briefly review and explain the so-called PIEO method [20, 21]. Let us trace back the original idea of Schrödinger quantization scheme, where the identification $i\frac{d}{dt} \leftrightarrow \hat{H}$ (Hamiltonian), so $i\frac{d}{dt}$ is named the Schrödinger operator in many references. Similarly, we have $(i\frac{d}{dt})^n \leftrightarrow \hat{H}^n$, now we set up the n-order differential equation for an operator $\hat{O}_e$

$$\left(i\frac{d}{dt}\right)^n \hat{O}_e = \lambda \hat{O}_e \tag{A.1}$$

when $n = 1$, it looks like in form to equation $i\frac{d}{dt}\psi = \hat{H}\psi$ ($\hbar = 1$). Thus, (A.1) is named n-order invariant eigenoperator equation with n-order eigenvalue. Using the Heisenberg equation $i\frac{d}{dt}\hat{O}_e = [\hat{O}_e, \hat{H}]$, ($\hbar = 1$). Equation (A.1) is rewritten as

$$\left(i\frac{d}{dt}\right)^n \hat{O}_e = [\cdots[[\hat{O}_e, \hat{H}], \hat{H}]\cdots, \hat{H}] = \lambda \hat{O}_e. \tag{A.2}$$

If such an $\hat{O}_e$ is found, $\sqrt[n]{\lambda}$ is called the energy-level gap. When for some Hamiltonian the n-fold commutator $[\cdots[[\hat{O}_e, \hat{H}], \hat{H}]\cdots, \hat{H}]$ in (A.2) is not proportional to $\hat{O}_e$, and it seems that the next $n+1$-fold commutator will not produce a constant multiplied by $\hat{O}_e$ either, then if there exists some state-vector space spanned by $|\phi\rangle_i$ in which the equation $[\cdots[[\hat{O}_e, \hat{H}], \hat{H}]\cdots, \hat{H}]|\phi\rangle_i = \lambda\hat{O}_e|\phi\rangle_i$ holds, in this limited space $\hat{O}_e$ is called the pseudo invariant eigenoperator (PIEO) of $\hat{H}$. Usually, $|\phi\rangle_i$ are the eigenvector of conservative quantities of the dynamic system which commute with the Hamiltonian. Now, the following equation

$$\left(i\frac{d}{dt}\right)^n \hat{O}_e\,|\phi\rangle_i = \lambda \hat{O}_e\,|\phi\rangle_i \tag{A.3}$$

may lead us to obtain some information of energy gap of the Hamiltonian.

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